Extreme Value Analysis of the Haezendonck–Goovaerts Risk Measure with a General Young Function[1]

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[1] Based on a joint work with Qihe Tang

Outline

1. Extreme risks and the HG risk measure
2. Brief introduction to Extreme Value Theory
3. Main results

Extreme risks

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Fisher–Tippett Theorem: Let $(X_1, X_2, \ldots, X_n)$ be a sequence of i.i.d. random variables and $M_n = \max\{X_1, \ldots, X_n\}$...

Choosing risk measures

Desired properties:

1. Coherence
   - monotonicity: $Y \leq X \Longrightarrow \rho(Y) \leq \rho(X)$
   - sub-additivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$
   - positive homogeneity: $\rho(\alpha X) = \alpha\rho(X)$ for $\alpha \geq 0$
   - translation invariance: $\rho(X+a) = \rho(X) + a$, for some certain amount a

2. Able to capture the tail behaviors of risks

Definition of HG risk measure

- X: risk variable
- q: confidence level between 0 and 1
- $\varphi(\cdot)$: normalized Young function, non-negative, convex on $[0,\infty)$, $\varphi(0)=0$, $\varphi(1)=1$ and $\varphi(\infty)=\infty$
- L_0^φ: the Orlicz heart, $L_0^\varphi = \{X : \mathrm{E}[\varphi(cX)] < \infty \text{ for all } c > 0\}$

Definition. Let h be the unique solution to the equation

$$\mathrm{E}\left[\varphi\left(\frac{(X-x)_+}{h}\right)\right] = 1-q.$$

Then the Haezendonck–Goovaerts risk measure (HG risk measure) for $X \in L_0^\varphi$ is defined as

$$H_q[X] = \inf_{x\in\mathbb{R}} (x+h) = x_* + h_*.$$

Properties

It was originally motivated from the Swiss premium principle and induced by the Orlicz norm.

For a convex Young function $\varphi(\cdot)$, the HG risk measure is a law invariant and coherent risk measure.

Consider the special case with $\varphi(t) = t$ for $t \in \mathbb{R}_+$. Then

$$H_q[X] = \inf_{x \in \mathbb{R}} \left(x + \frac{\mathrm{E}\left[(X-x)_+\right]}{1-q} \right) = \frac{1}{1-q} \int_q^1 \mathrm{VaR}_p[X] \mathrm{d}p,$$

and, thus, the HG risk measure is reduced to the well-known Tail Value-at-Risk (TVaR).

Literature review

- Haezendonck and Goovaerts (1982, IME)
- Goovaerts, Kaas, Dhaene and Tang (2004, IME)
- Bellini and Rosazza Gianin (2008a, J. of Banking and Finance; 2008b, Stat. Decis.; 2012, IME)
- Nam, Tang and Y. (2011, IME)
- Tang and Y. (2012, IME)
- Goovaerts, Linders, Van Weert and Tank (2012, IME)
- Mao and Hu (2012, IME)
- Ahn and Shyamalkumar (2014, IME)

Computation

Emphasize the tail areas; Solvency II sets the confidence level of VaR to 0.995.

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The HG risk measure does not have an explicit expression. A common approach is to do simulations, but

- simulations do not help us to qualitatively understand the tail behavior of a risk;
- simulations are not quite efficient when the confidence level is high.

Asymptotics

We derive asymptotics as an alternative way to study risk measures.

Asymptotics are equivalent expressions of the risk measure as the confidence level is very close to 1.

- Asymptotic expressions provide us insights.
- Asymptotic expressions are very easy to compute and it takes almost no time to get the results.

We shall focus on the asymptotic behavior of $H_q[X]$ as the confidence level $q \uparrow 1$.

Tang and Yang (2012, IME)

We considered a power Young function, $\varphi(t) = t^k$, for $k \geq 1$, and derived the following:

- for the Fréchet case: $H_q[X] \sim c_1 F^{\leftarrow}(q)$
- for the Gumbel case:
$$\begin{cases} H_q[X] \sim F^{\leftarrow}(1 - c_2 q), & \text{when } \hat{x} = \infty \\ \hat{x} - H_q[X] \sim (\hat{x} - F^{\leftarrow}(1 - c_2 q)), & \text{when } \hat{x} < \infty \end{cases}$$
- for the Weibull case: $\hat{x} - H_q[X] \sim c_3 (\hat{x} - F^{\leftarrow}(q))$

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- Convergence of maxima — EVT

Consider a sequence of i.i.d. random variables $(X_1, X_2, ..., X_n)$ with the distribution function F. Denote $M_n = \max\{X_1, X_2, ..., X_n\}$ the block maxima.

The central result of EVT studies how the df of the normalized M_n converges.

Fisher–Tippett theorem

A df F is said to belong to the max-domain of attraction (MDA) of a df G, denoted by $F \in \mathrm{MDA}(G)$, if

$$\lim_{n\to\infty} \Pr\left(\left(M_n - d_n\right)/c_n \le x\right) = G(x)$$

holds for some norming constants $c_n > 0$ and $d_n \in \mathbb{R}$, $n \in \mathbb{N}$.

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By the classical Fisher–Tippett theorem (see Fisher and Tippett (1928) and Gnedenko (1943, Ann. of Math.)), G has to be the generalized extreme value (GEV) distribution, whose standard structure is given by

$$G_\gamma(x) = \exp\left\{-\left(1+\gamma x\right)^{-1/\gamma}\right\}, \qquad 1+\gamma x > 0, \gamma \in \mathbb{R},$$

where for $\gamma = 0$ the right-hand side is $\exp\left\{-\mathrm{e}^{-x}\right\}$.

Extended regular variation

Definition A positive measurable function $f(\cdot)$ is said to be extended regularly varying with index $\gamma \in \mathbb{R}$, denoted by $f(\cdot) \in \mathrm{ERV}_\gamma$, if there exists an auxiliary function $a(\cdot) > 0$ such that, for all $y > 0$,

$$\lim_{x \to \infty} \frac{f(xy) - f(x)}{a(x)} = \frac{y^\gamma - 1}{\gamma}.$$

When $\gamma = 0$, the right-hand side is interpreted as $\log y$.

The auxiliary function $a(\cdot)$ is often chosen to be

$$a(x) = \begin{cases} \gamma f(x), & \gamma > 0, \\ f(x) - x^{-1} \int_0^x f(s)\mathrm{d}s, & \gamma = 0, \\ -\gamma(f(\infty) - f(x)), & \gamma < 0. \end{cases}$$

MDA of the GEV distribution

Define $U(\cdot)$ as the quantile/inverse function of $1/\overline{F}$,

$$U(t) = \left(\frac{1}{\overline{F}}\right)^{\leftarrow}(t) = F^{\leftarrow}\left(1 - \frac{1}{t}\right).$$

$F \in \mathrm{MDA}(G_\gamma)$ if and only if $U \in \mathrm{ERV}_\gamma$, where

$$G_\gamma = \begin{cases} \Phi_{1/\gamma}, & \gamma > 0 \quad \text{(Fréchet)}, \\ \Lambda, & \gamma = 0 \quad \text{(Gumbel)}, \\ \Psi_{-1/\gamma}, & \gamma < 0 \quad \text{(Weibull)}. \end{cases}$$

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Challenges of the problem

Recall the definition of the HG risk measure.

$$\mathrm{E}\left[\varphi\left(\frac{(X-x)_+}{h}\right)\right] = 1-q.$$

The HG risk measure $H_q[X] = \inf_{x\in\mathbb{R}}(x+h)$.

For power Young functions in the previous section:

$$h = \left(\frac{\mathrm{E}\left[(X-x)_+^k\right]}{1-q}\right)^{1/k}.$$

However, for a general Young function in this section:

We have to deal with the implicit function of h throughout the work.

Assumptions

- Assmptions for the Young function $\varphi(\cdot)$:
  - $\varphi(\cdot) \in \mathrm{RV}_\alpha(0+) \cap \mathrm{RV}_\beta(\infty)$ for some $1 < \alpha, \beta < \infty$
  - strictly convex and continuously differentiable in $[0, \infty)$
  - $\varphi'_+(0) = 0$
- Assmptions for the risk variable X:
  - $X \in L_0^\varphi$
  - $F \in \mathrm{MDA}(G_\gamma)$ with $-\infty < \gamma < \alpha^{-1} \wedge \beta^{-1}$

Main Result

Define a positive random variable Y distributed by

$$\Pr(Y \le y) = 1 - (1 + \gamma y)^{-1/\gamma}$$

for all $y > 0$ such that $1 + \gamma y > 0$.

Let k be the unique positive solution of the equation

$$\mathrm{E}\left[\varphi'(kY)\right] = \mathrm{E}\left[\varphi'(kY)\,kY\right].$$

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As $q \uparrow 1$, the HG risk measure is given by $H_q[X] = x_* + h_*$, where

$$\overline{F}(x_*) \sim \frac{1-q}{\mathrm{E}\left[\varphi(kY)\right]} \quad \text{and} \quad h_* \sim \frac{a(1/\overline{F}(x_*))}{k}.$$

The Fréchet case

Corollary As $q \uparrow 1$,

(i) $\gamma > 0$:

$$H_q[X] \sim \left(1 + \frac{\gamma}{k}\right) \left(\int_0^{\infty} \left(1 + \frac{\gamma}{k} z\right)^{-1/\gamma} \mathrm{d}\varphi(z) \right)^{\gamma} \mathrm{VaR}_q[X];$$

The Gumbel case

(ii) $\gamma = 0$:

if $\hat{x} = \infty$ then

$$H_q[X] \sim F^{\leftarrow}\left(1 - \frac{1-q}{\int_0^\infty \mathrm{e}^{-z/k}\mathrm{d}\varphi(z)}\right),$$

while if $\hat{x} < \infty$ then

$$\hat{x} - H_q[X] \sim \hat{x} - F^{\leftarrow}\left(1 - \frac{1-q}{\int_0^\infty \mathrm{e}^{-z/k}\mathrm{d}\varphi(z)}\right);$$

The Weibull case

(iii) $\gamma < 0$:

$$\hat{x} - H_q[X] \sim \left(1 + \frac{\gamma}{k}\right) \left(\int_0^{-k/\gamma} \left(1 + \frac{\gamma}{k} z\right)^{-1/\gamma} \mathrm{d}\varphi(z) \right)^{\gamma} (\hat{x} - \mathrm{VaR}_q[X]) .$$

Key steps

- $H_q[X] = x_* + h_*$
- $a(\cdot)$: the auxiliary function
- $t_* = 1/\overline{F}(x_*)$

Step 1. As $q \uparrow 1$, we have $x_* \uparrow \hat{x}$.

Step 2. As $q \uparrow 1$, we have $a(t_*) \asymp h_*$.

Step 3. $\lim_{q\uparrow 1} a(t_*)/h_* = k$.

An example with exact solution for comparison

In order to get the exact value of $H_q[X]$, we choose the Young function as

$$\varphi(t) = \frac{t^{2.2} + t^{1.1}}{2}.$$

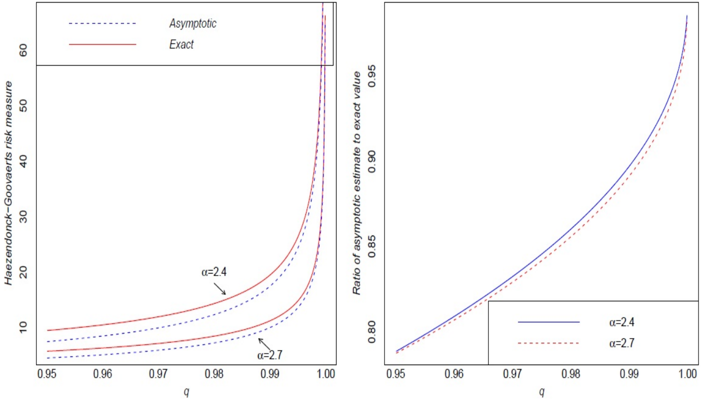
By the quadratic formula, we can solve h as

$$h = \left(\frac{\mathrm{E}\left[(X-x)_+^{1.1}\right] + \sqrt{\left(\mathrm{E}\left[(X-x)_+^{1.1}\right]\right)^2 + 8(1-q)\mathrm{E}\left[(X-x)_+^{2.2}\right]}}{4(1-q)} \right)^{1/1.1}.$$

Solve x_* from the equation $h'(x_*) = -1$.

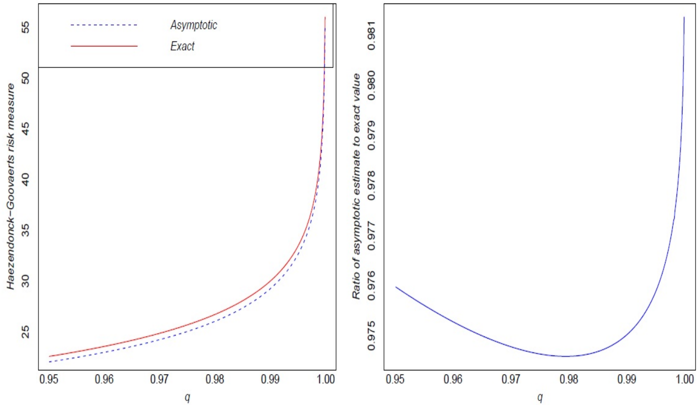
The Fréchet case

Graph 1. $F = \text{Pareto}(\alpha = 2.4 \;\&\; 2.7, \theta = 1)$



The Gumbel case

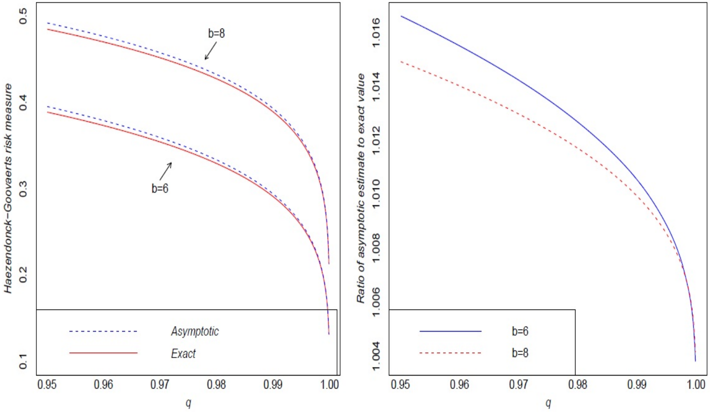
Graph 2. $F = \text{Lognormal}(\mu = 2, \sigma = 0.5)$



Numerical Examples

The Weibull case

Graph 3. $F = \mathrm{Beta}(a = 2, b = 6 \;\&\; 8)$



Thank you very much for your attention!